④ Stability of persistence barcodes / diagrams:

We have an operator $\mathrm{Dgm} : f \longmapsto \mathrm{Dgm}(f)$

real-valued function

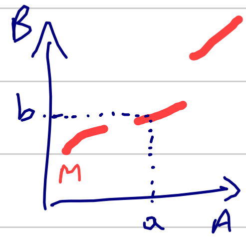
persistence diagram of the filtration of sublevel sets of f (whenever defined)

↳ we want to show that Dgm is Lipschitz continuous:

- metric on functions $f, g : X \to \mathbb{R}$ (for X compact): $\|f-g\|_\infty$
- metric on persistence diagrams: <u>bottleneck distance</u> d_∞.

$\boxed{\text{Def:}}$ A <u>partial matching</u> $A \leftrightarrow B$ is a subset $M \subseteq A \times B$ such that the natural projections $\pi_A : A\times B \to A$ and $\pi_B : A \times B \to B$, restricted to M, are injective.

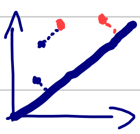
↳ $\begin{cases} \forall a \in B, \exists \leq 1 \ b \in B \text{ s.t. } (a,b) \in M \\ \forall b \in B, \exists \leq 1 \ a \in A \text{ s.t. } (a,b) \in M \end{cases}$



$\boxed{\text{Def:}}$ The <u>p-th cost</u> $c_p(M)$ of a partial matching $M : A \leftrightarrow B$ is given by:

(for $A, B \subseteq \mathbb{R}^2$)

$$c_p(M) := \left(\sum_{(a,b)\in M} \|a-b\|_\infty^p + \sum_{\substack{c \notin \pi_A|_M \\ c \notin \pi_B|_M}} \left(\frac{c_y - c_x}{2}\right)^p \right)^{1/p}.$$



$\boxed{\text{Def:}}$ The <u>p-th distance</u> between $A, B \subseteq \mathbb{R}^2$ is:

$$d_p(A,B) := \inf_{M : A \leftrightarrow B} c_p(M).$$

Note: The bottleneck distance d_∞ is obtained by letting $p \to \infty$ in the above definitions (sums are replaced by max).

Thm: (Stability) [Cohen-Steiner, Edelsbrunner, Harer 2005]
For any pfd functions $f, g : X \to \mathbb{R}$ (i.e. whose sublevel sets have finite-dimensional homology):

$$d_\infty(\mathrm{Dgm}\, f, \mathrm{Dgm}\, g) \leq \|f - g\|_\infty .$$

Def: An ε-interleaving between persistence modules M, N over $\mathbb{R}$ is given by two families of linear maps:

$$\forall t \in \mathbb{R}, \begin{cases} \varphi_t : M_t \longrightarrow N_{t+\varepsilon} \\ \psi_t : N_t \longrightarrow M_{t+\varepsilon} \end{cases}$$

such that the following diagrams commute $\forall s \leq t \in \mathbb{R}$:

$$\begin{tikzcd} M_s \arrow[r, "m_s^t"] \arrow[d, "\varphi_s"'] & M_t \arrow[d, "\varphi_t"] \\ N_{s+\varepsilon} \arrow[r, "n_{s+\varepsilon}^{t+\varepsilon}"'] & N_{t+\varepsilon} \end{tikzcd}$$

φ, ψ are morphisms between shifted persistence modules

$$\begin{tikzcd} M_{s+\varepsilon} \arrow[r, "m_s^t"] & M_{t+\varepsilon} \\ N_s \arrow[u, "\psi_s"] \arrow[r, "n_s^t"'] & N_t \arrow[u, "\psi_t"'] \end{tikzcd}$$

$$\begin{tikzcd} M_s \arrow[rr, "m_s^{s+2\varepsilon}"] \arrow[dr, "\varphi_s"'] & & M_{s+2\varepsilon} \\ & N_{s+\varepsilon} \arrow[ur, "\psi_{s+\varepsilon}"'] & \end{tikzcd}$$

φ, ψ compose to shifted identities

$$\begin{tikzcd} & M_{s+\varepsilon} \arrow[dr, "\varphi_{s+\varepsilon}"] & \\ N_s \arrow[ur, "\psi_s"] \arrow[rr, "n_s^{s+2\varepsilon}"'] & & N_{s+2\varepsilon} \end{tikzcd}$$

Note: A 0-interleaving is an isomorphism of persistence modules.

Def: The <u>interleaving distance</u> d_i on persistence modules is defined as:

$$d_i(M,N) := \inf\{\varepsilon \geq 0 \mid M, N \text{ are } \varepsilon\text{-interleaved}\}$$

Thm: (Isometry) [Chazal et al. 2009, Lesnick 2011]
For any pfd persistence modules M, N,

$$d_i(M,N) = d_\infty(\operatorname{Dgm} M, \operatorname{Dgm} N).$$

Thus, persistence barcodes/diagrams are complete algebraic and metric descriptors of persistence modules.

$\boxed{Q}$ How relevant is the interleaving distance d_i on the space of persistence modules?
↳ By the Stability and Isometry theorems, we have $d_i(H_*(F), H_*(G)) \leq \|f-g\|_\infty$ for all pfd functions $f, g : X \to \mathbb{R}$ and their associated sublevel sets filtrations F, G. In fact:

Prop: For any functions $f, g : X \to \mathbb{R}$ (possibly not pfd) and their associated sublevel sets filtrations F, G,

$$d_i(H_*(F), H_*(G)) \leq \|f-g\|_\infty.$$

→ <u>proof:</u> Let $\varepsilon = \|f-g\|_\infty$. Then, $\forall t \in \mathbb{R}$ we have

$$F_t \subseteq G_{t+\varepsilon} \quad \text{and} \quad G_t \subseteq F_{t+\varepsilon}.$$

Hence the (commutative) diagrams of inclusions $\forall s \leq t \in \mathbb{R}$:

$$\begin{tikzcd} F_s \arrow[r] \arrow[d] & F_t \arrow[d] \\ G_{s+\varepsilon} \arrow[r] & G_{t+\varepsilon} \end{tikzcd} \qquad \begin{tikzcd} F_s \arrow[rr] \arrow[dr] & & F_{s+2\varepsilon} \\ & G_t \arrow[ur] & \end{tikzcd}$$

and reciprocally. The commutative diagrams of ε-interleaving between $H_*(F)$ and $H_*(G)$ follow then from the functoriality of H_*. $\blacksquare$

$\boxed{\text{Thm:}}$ [Lesnick 2011]

d_i satisfies the following universality property on the space of persistence modules:

For any metric d such that $d(H_*(F), H_*(G)) \leq \|f-g\|_\infty$ for all functions $f, g : X \to \mathbb{R}$, one has $d \leq d_i$.

Thus, d_i is the most discriminative among all stable metrics on the space of persistence modules.

(in the sense that the operator $f \mapsto H_*(F)$ is 1-Lipschitz)